Spin Wave Theory of Spin 1/2 XY Model with Ring Exchange on a Triangular Lattice

Solomon A. Owerre
[1] *Groupe de physique des particules, Département de physique, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, Québec, Canada, H3C 3J7*
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We present the linear spin wave theory calculation of the superfluid phase of a hard-core boson J-K model with nearest neighbour exchange J and four-particle ring-exchange K at half filling on the triangular lattice, as well as the phase diagrams of the system at zero and finite temperatures. We find that the pure J model (XY model) which has a well known uniform superfluid phase with an ordered parameter $M_x = \langle S_i^x \rangle \neq 0$ at zero temperature is quickly destroyed by the inclusion of a negative-K ring-exchange interactions, favouring a state with a $\left(\frac{4\pi}{3}, 0\right)$ ordering wavevector. We further study the behaviour of the finite-temperature Kosterlitz-Thouless phase transition (T_{KT}) in the uniform superfluid phase, by forcing the universal quantum jump condition on the finite-temperature spin wave superfluid density. We find that for $K<0$, the phase boundary monotonically decreases to $T = 0$ at $K/J = -4/3$, where a phase transition is expected and T_{KT} decreases rapidly while for positive K, T_{KT} reaches a maximum at some $K \neq 0$. It has been shown on a square lattice using quantum Monte Carlo(QMC) simulations that for small $K>0$ away from the XY point, the zero-temperature spin stiffness value of the XY model is decreased[6]. Our result seems to agree with this trend found in QMC simulations.

I. INTRODUCTION

The effective studies of continuum field theories have resulted in detailed predictions for the low-energy physics of quantum spin systems in two dimensions. Spin wave theory can provide us with a rather accurate picture of the low-lying states of quantum spin systems. There are several versions of spin wave theory. The standard spin wave theory is based on Holstein-Primakoff representation[10] which was first applied to the study of Heisenberg model by Anderson[4] and further extended to second order by Kubo[5] and Oguchi[9].

Spin wave theory was thought to be unsatisfactory in the case of XY model until Gomez-Santos and Joannopoulous[1] showed that by a good choice of quantization axis, one can obtain a good theoretical result for XY model. Since then numerous applications of spin wave theory have been carried out on XY model with different lattice configurations and the results obtained so far are in a good agreement with quantum Monte Carlo simulations(QMC)[3,12].

Another interesting area is the multiple(ring) spin-exchange models. This model was first introduced to describe the magnetic properties of solid ^{3}He[15]. It incorporates ring-exchange interactions over plaquettes such as spin 1/2 four-spin XY ring exchange of the form:

$$H_K = -K \sum_{\langle ijkl \rangle} \left(S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+ \right), \qquad (1)$$

where the summation runs over plaquettes with the indices running counter-clockwise, i and j are nearest neighbours lying opposite to k and l.

The ring exchange interaction is important in Wigner crystal near the melting density[13]. This model, alone or in competition with pure XY model with nearest neighbour exchange has attracted considerable attention over the years and has been studied extensively on a square lattice using a stochastic series expansion (SSE) quantum Monte Carlo method[6] also a comprehensive theoretical study (spin wave theory) has been done on a square lattice[8]. It has been suggested in recent works that models of this form may harbour exotic ground state properties, including de-confined quantum critical points or quantum spin liquid phases.

In this paper, we shall calculate the exact (numerical) value of the Kosterlitz-Thouless temperature, superfluid density and other low temperature thermodynamical properties of spin 1/2 XY model with ring exchange interaction (1) on a triangular lattice using linear spin wave theory without the inclusion of the repulsive interaction between bosons.

The format of the paper is as follows: In Sec.II, we present the model Hamiltonian. In Sec.III, we apply linear wave theory by choosing our quantization axis along the x direction and use it to diagonalize the Hamiltonian and obtain its energy. In Sec.IV, we analyze the dispersion and plot it for some values of K/J. In Sec.V, we explore the zero temperature superfluid density using the diagonalized Hamiltonian and plot it for some values of K/J. In Sec.VI, we calculate the finite temperature superfluid density and the value of the Kosterlitz-Thouless temperature for this model and in Sec. VII, we make some concluding remarks.

II. MODEL

In this section, we shall present the model Hamiltonian and the mean field theory argument of the J-K model on a triangular lattice, similar work was done on a square

lattice[8]. Our model Hamiltonian is given by

$$\begin{aligned} H = H_J + H_K = &-J\sum_{\langle ij\rangle}\left(S_i^+S_j^- + S_i^-S_j^+\right) \\ &-K\sum_{\langle ijkl\rangle}\left(S_i^+S_j^-S_k^+S_l^- + S_i^-S_j^+S_k^-S_l^+\right). \end{aligned} \tag{2}$$

The first summation is over nearest neighbour pairs on a triangular lattice and the second summation runs over the three possible plaquette orientations on a triangular lattice as shown in Fig.(1). It has been shown that this Hamiltonian undergoes a Kosterlitz-Thouless phase transition for $K = 0$ at $T_{KT} \approx 0.69$ for 2D model and a superfluid phase for temperatures less than T_{TK}[16]. For $J > 0, K > 0$ the Hamiltonian leads to an in-plane quantum ferromagnet and for $K < 0$, there is a sign problem which prevents (QMC) simulations[6] though it is not possible to capture this sign problem in the linear spin wave theory. QMC simulation on a square lattice has been performed for $K > 0$ in which there is no sign problem[17]. Lets define the spins as classical vectors by making the transformation $S_i^- = \rho e^{i\phi_i}$. The model Hamiltonian becomes

$$\begin{aligned} H = &-2J\sum_{\langle ij\rangle}\rho^2\cos\left(\phi_i - \phi_j\right) \\ &-2K\sum_{\langle ijkl\rangle}\rho^4\cos\left(\phi_i - \phi_j + \phi_k - \phi_l\right). \end{aligned} \tag{3}$$

Now lets consider the case $J, K > 0$, in this case, the minimum energy permits us to have $\phi_i = \phi_j$ for the J term and $\phi_i = \phi_j$, $\phi_k = \phi_l$ for the K term which leads to a ferromagnetic ordering of spins. Consider the case $J, K<0$, in this case, the minimum energy permits us to have $\phi_i - \phi_j = \pi$ or $\phi_i = 0, \phi_j = \pi$ for the J term and $\phi_i - \phi_j + \phi_k - \phi_l = 0$ or $\phi_i = \phi_j = \phi_k = 0, \phi_l = \pi$ for the K term which leads to ud (up-down) state for the J term and uuud state on the plaquettes, basically we have two configurations of spins on the lattice.

This model has also been considered by L. Balents and A. Paramekanti[14] with the inclusion of the repulsive interaction U term between bosons and also in the regime where $J << K$. They showed that when $J = 0$, the four-spin exchange leads to a manifold of ground states with gapless excitations and critical power-law correlations and when $J \neq 0$, fluctuations select a four-fold ferrimagnetically ordered ground state with a small spin (superfluid) stiffness which breaks the global $U(1)$ and translational symmetry but they did not obtain any exact (numerical) value of the Kosterlitz-Thouless temperature and the superfluid density.

From their Hamiltonian they argued that with $J = 0$, the ground state is independent of the sign of K and for non-zero J, the sign of K is vital, for $J < 0$ with $K < 0$, there is a ferromagnetic phase while $J > 0$ leads to a $\sqrt{3}\times\sqrt{3}$ Neel order which are also the ground states for large $|J/K|$. They finally concluded that there are no phase transitions at any non-zero J other than the well known phases.

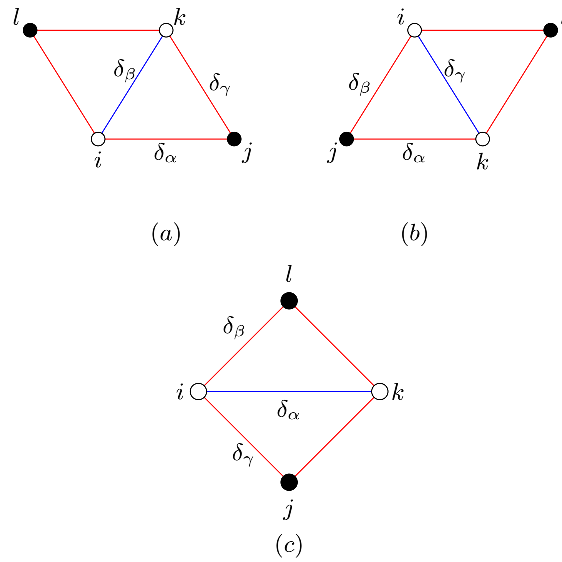

FIG. 1: (Color online) The three plaquette orientations (a), (b), and (c) on a triangular lattice with the position coordinates $\delta_\alpha = (1, 0)$, $\delta_\beta = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\delta_\gamma = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

III. LINEAR SPIN WAVE THEORY

The basic assumption of spin wave theory lies on selecting a classical ground state and determining the fluctuation around it. In other words, one considers quantum fluctuations very close to an ordered ground state configuration of the system under study. By the usual mapping between spins $S = 1/2$ and the hard-core bosons, we can view (2) as a hard-core boson model. For $J >> K$ or $K = 0$, the $T = 0$ ground state (in-plane ferromagnet) has an ordered parameter $M_x = \langle S_i^x\rangle \neq 0$ which breaks the $U(1)$ global rotational symmetry or a superfluid phase in the hard-core boson version[18–20]. One can therefore perform a spin wave expansion around this ordered state configuration by introducing the boson operators a_i and $a_i^\dagger$ which represent the low-energy spin wave excitations out of $\langle S_i^x\rangle$ and treat other terms in (2) as perturbations.

We shall follow the procedure of Gomez-Santos and Joannopoulos[1] and choose our quantization axis along the x direction (instead of z direction). This allows us to write the Holstein-Primakoff representation[10] in the linear spin wave theory as

$$\begin{aligned} S_i^x &= \frac{1}{2} - a_i^\dagger a_i, \\ S_i^y &\approx \frac{1}{2i}\left(a_i^\dagger - a_i\right). \end{aligned} \tag{4}$$

The method for diagonalizing the Hamiltonian (2) is given by R. Schaffer *et al* [8]. It involves writing (2) in terms of S_j^x and S_j^y using $S_j^\pm = S_j^x \pm iS_j^y$, after which we substitute (4) into the resulting equation and then Fourier transform taking into account the summation

over all the three plaquettes for the K-term and finally diagonalize using Bogoliubov transformation. After all these mathematical gymnastics, the diagonalized Hamiltonian is

$$\begin{aligned} H = H_{MF} &+ \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} - A_{\mathbf{k}}\right) \\ &+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\alpha^{\dagger}_{\mathbf{k}}\alpha_{\mathbf{k}} + \alpha^{\dagger}_{-\mathbf{k}}\alpha_{-\mathbf{k}}\right). \end{aligned} \tag{5}$$

Here, the mean-field energy and the coefficients are totally different from those obtained on a square lattice[8], they are given by

$$H_{MF} = -3\left(\frac{1}{2}JN + \frac{1}{8}KN\right), \tag{6}$$

$$\omega_{\mathbf{k}} = \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}, \tag{7}$$

$$A_{\mathbf{k}} = JQ_{\mathbf{k}} + KR_{\mathbf{k}}, \tag{8}$$

$$B_{\mathbf{k}} = JS_{\mathbf{k}} + KT_{\mathbf{k}}, \tag{9}$$

where

$$Q_{\mathbf{k}} = 3\left(1 - \frac{\gamma_{\mathbf{k}}}{2}\right), \tag{10}$$

$$S_{\mathbf{k}} = \frac{3}{2}\gamma_{\mathbf{k}}, \tag{11}$$

$$R_{\mathbf{k}} = 3\left\{\frac{1}{2} - \frac{1}{2}\gamma_{\mathbf{k}} + \frac{1}{8}\left(\gamma_{\mathbf{k}} + \bar{\gamma_{\mathbf{k}}}\right)\right\}, \tag{12}$$

$$T_{\mathbf{k}} = 3\left\{\frac{1}{2}\gamma_{\mathbf{k}} - \frac{1}{8}\left(\gamma_{\mathbf{k}} + \bar{\gamma_{\mathbf{k}}}\right)\right\}, \tag{13}$$

and the lattice structure constants are given by

$$\begin{aligned} \gamma_{\mathbf{k}} &= \frac{1}{3}\left(\cos k_x + 2\cos\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), \\ \bar{\gamma_{\mathbf{k}}} &= \frac{1}{3}\left(\cos\sqrt{3}k_y + 2\cos\frac{3k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right). \end{aligned} \tag{14}$$

We can see that the Hamiltonian (5) reduces to pure XY model in the limit $K = 0$ as expected. This diagonalized Hamiltonian (5) will be used to analyse some properties such as dispersion, ground state, internal energy, superfluid density etc., all as a function of K/J.

IV. DISPERSION

We shall now study the dispersion of J-K model as a function of K/J. Similar to J model, the dispersion is given by

$$\epsilon\left(\mathbf{k}\right) = 2\omega_{\mathbf{k}} = 2\sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}, \tag{15}$$

but in this case the coefficients $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are given by (8) and (9). The graph of (15) is shown in Fig.(3) for specific values of K/J. For $K/J = -4/3$, the dispersion takes the compact form

$$\epsilon\left(\mathbf{k}\right) = \sqrt{1 - \bar{\gamma_{\mathbf{k}}}}, \tag{16}$$

where $\bar{\gamma_{\mathbf{k}}}$ is given by (14). For $K/J = -2$, the dispersion $\epsilon\left(\mathbf{k}\right) = \sqrt{0} = 0$ for all values of $\mathbf{k}$ and no reasonable plot can be obtained below this value of K/J because the dispersion develops an imaginary part along the k_x and k_y directions. The dispersion is also zero at $\mathbf{k} = (0, 0)$ for all values of K/J. At $\mathbf{k} = (\frac{4\pi}{3}, 0), (\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}})$ which corresponds to the corners of Brillouin zone Fig.(4), it is given by

$$\epsilon\left(\mathbf{k}\right) = \frac{3}{4}\sqrt{24 + 30(K/J) + (3K/J)^2}, \tag{17}$$

which clearly vanishes at two points $K/J = -4/3$ and $K/J = -2$. The plot of (17) is shown in Fig.(2). There are two regions I and II, region I corresponds to $K/J \geq -4/3$ and region II corresponds to $K/J \leq -2$. Since the dispersion (15) develops an imaginary part along the k_x and k_y directions for $K/J < -2$, we see that region II is not physical.

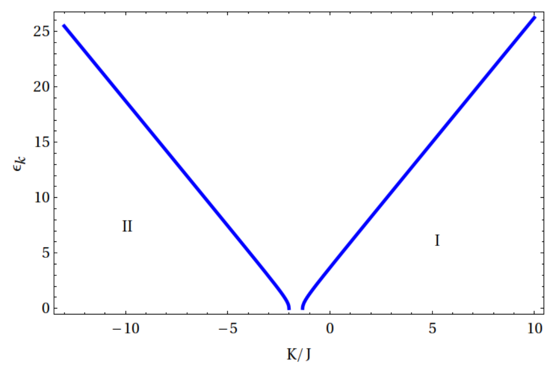

FIG. 2: (Color online) The dispersion $\epsilon\left(\mathbf{k}\right)$ as a function of K/J at the corners of the Brillouin zone

The dispersion along the k_x direction is also shown in Fig.(3) for several values of K/J. In this case the dispersion shows a linear function of k_x and a zero mode at the center of the Brillouin zone $\mathbf{k}$=$\mathbf{Q}$=$(0, 0)$ except for $K/J = -4/3$ where it develops three zero modes (minima) at $\mathbf{k}= \pm\mathbf{Q} =\left(\pm\frac{4\pi}{3}, 0\right)$ and $\mathbf{k}= \mathbf{Q}$=$(0, 0)$. If one chooses ferromagnetic ordering along the k_y axis then the corresponding ordering wave vector is $\mathbf{k}= \pm\mathbf{Q} =\left(\pm\frac{4\pi}{3}, 0\right)$ which is the soft modes of the dispersion for $K/J = -4/3$. The linear spin wave instability of the excitation spectrum at the corner of the Brillioun zone $\mathbf{k}= \pm\mathbf{Q} =\left(\pm\frac{4\pi}{3}, 0\right)$ occurs for $K/J = -4/3$. For the pure XY model $K/J = 0$, there is a gapless excitation at $\mathbf{Q}$ $=(0, 0)$ (Goldstone mode of the superfluid phase), but there is no minima at the ordering wave vector. Near the zero modes for $K/J = -4/3$ we have

$$\epsilon(\mathbf{k}) = \frac{\sqrt{3}}{2}\left(k_x^2 + k_y^2\right)^{1/2} = \sqrt{3}Jk. \tag{18}$$

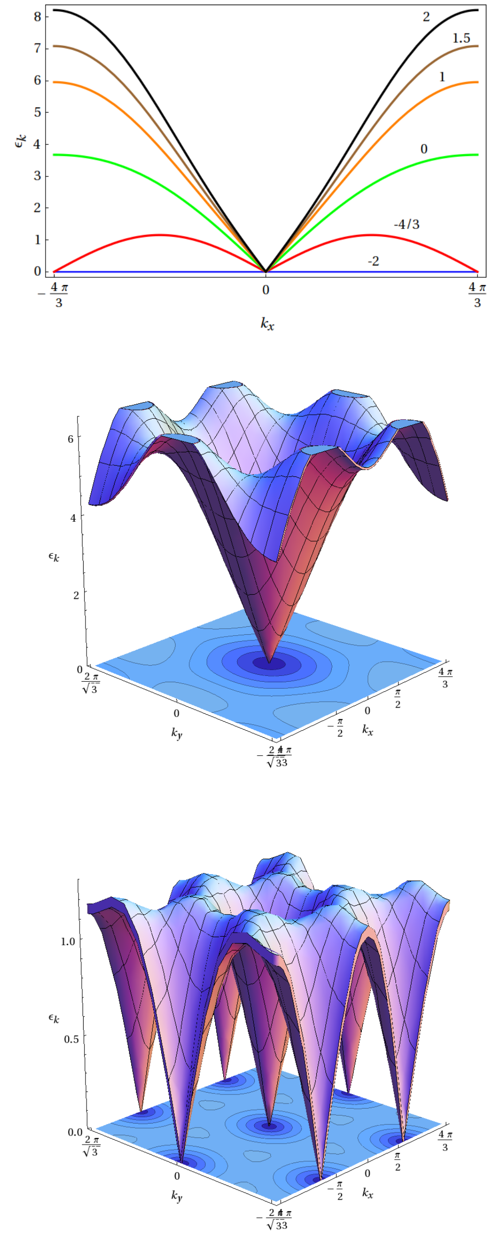

FIG. 3: (Color online) The dispersion $\epsilon(\mathbf{k})$ along k_x direction for several values of K/J (first), the dispersion $\epsilon(\mathbf{k})$ as a function of $\mathbf{k} = (k_x, k_y)$, for $K/J = 4/3$ (second) and $K/J = -4/3$ (third).

V. SUPERFLUID DENSITY AT $T = 0$

In this section we shall explore the superfluid density of the full J-K model on a triangular lattice. In this case we must consider the influence of the ring exchange term on the applied phase twist θ. The relevant transformations are similar to that of a square lattice[8], with this transformations the ring exchange term becomes

$$S_i^+ S_j^- S_k^+ S_l^- = S_i^+ S_j^- S_k^+ S_l^- e^{i(\theta_i + \theta_k - \theta_j - \theta_l)}, \tag{19}$$

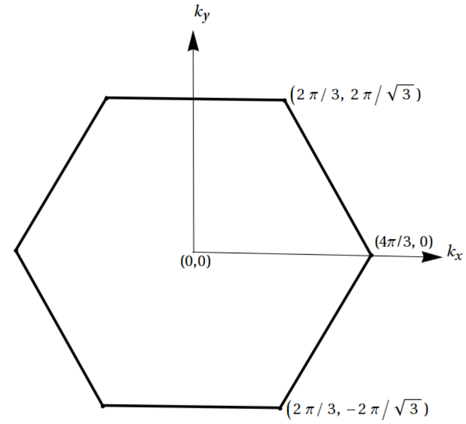

FIG. 4: (Color online) The Brillouin zone of the triangular lattice.

using the labeling in Fig.(1), we have $\theta_i = \theta_l$, $\theta_j = \theta_k$, therefore the θ dependence of the ring exchange terms cancel. The twisted J-K Hamiltonian becomes

$$\begin{aligned} H(\theta) = &-2J \sum_{\langle ij \rangle} \left\{ \left(S_i^x S_j^x + S_i^y S_j^y \right) \cos\theta \right. \\ &\left. + \left(S_i^x S_j^y - S_i^y S_j^x \right) \sin\theta \right\} \\ &- K \sum_{\langle ijkl \rangle} \left(S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+ \right). \end{aligned} \tag{20}$$

In the linear spin wave theory (20) reduces to

$$\begin{aligned} H(\theta) = &-2J \sum_{\langle ij \rangle} \left(S_i^x S_j^x + S_i^y S_j^y \right) \cos\theta \\ &- K \sum_{\langle ijkl \rangle} \left(S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+ \right). \end{aligned} \tag{21}$$

Notice that the coefficients of $\sin\theta$ vanishes in the case. Comparing (2) and (21), we see that the effect of the twist is to rescale the nearest-neighbour exchange interaction in (2) as $J \rightarrow J\cos\theta$. Therefore, the diagonalized form of (21) is exactly of the form (5) with J replaced by $J\cos\theta$. That is

$$\begin{aligned} H(\theta) = &H_{MF}(\theta) + \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}}(\theta) - A_{\mathbf{k}}(\theta) \right) \\ &+ \sum_{\mathbf{k}} \omega_{\mathbf{k}}(\theta) \left(\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + \alpha_{-\mathbf{k}}^\dagger \alpha_{-\mathbf{k}} \right), \end{aligned} \tag{22}$$

where the twisted mean-field energy and the coefficients are given by

$$H_{MF}(\theta) = -3\left(\frac{1}{2} JN\cos\theta + \frac{1}{8} KN \right), \tag{23}$$

$$\omega_{\mathbf{k}}(\theta) = \sqrt{A_{\mathbf{k}}(\theta)^2 - B_{\mathbf{k}}(\theta)^2}, \tag{24}$$

$$\begin{aligned} A_{\mathbf{k}}(\theta) &= JQ_{\mathbf{k}}\cos\theta + KR_{\mathbf{k}}, \\ B_{\mathbf{k}}(\theta) &= JS_{\mathbf{k}}\cos\theta + KT_{\mathbf{k}}, \end{aligned} \tag{25}$$

the coefficients $Q_{\mathbf{k}}, R_{\mathbf{k}}, S_{\mathbf{k}}$, and $T_{\mathbf{k}}$ remain the same as (11)–(13).

The free energy is given by

$$F(\theta) = -\frac{1}{\beta}\ln Z(\theta) = E_0(\theta) + \frac{1}{\beta}\ln\left(1 - e^{-\beta\epsilon_{\mathbf{k}}(\theta)}\right), \quad (26)$$

where the ground state energy $E_0(\theta)$ is given

$$E_0(\theta) = H_{MF}(\theta) + \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}}(\theta) - A_{\mathbf{k}}(\theta)\right). \quad (27)$$

At zero temperature $\beta = \infty$, the free energy is simply the ground state energy. We shall calculate the superfluid density from the first principle. Taylor expanding the ground state energy we have

$$E_0(\theta) = E_0(\theta = 0) + \frac{1}{2}\rho_s\theta^2 + O(\theta^4), \quad (28)$$

$$\rho_s(T = 0) = \frac{1}{N}\frac{\partial^2 E_0(\theta)}{\partial\theta^2}|_{\theta=0}. \quad (29)$$

From (27) and (29) we obtain

$$\begin{aligned}\rho_s(T = 0) \;=\; & \frac{3J}{4} + \frac{J}{2N}\sum_{\mathbf{k}}\Big[Q_{\mathbf{k}} - \frac{1}{\omega_{\mathbf{k}}} \quad (30)\\ & \cdot\;\left(J\left(Q_{\mathbf{k}}^2 - S_{\mathbf{k}}^2\right) + K\left(Q_{\mathbf{k}}R_{\mathbf{k}} - S_{\mathbf{k}}T_{\mathbf{k}}\right)\right)\Big].\end{aligned}$$

Note that we have divided by 2 to account for the dimensionality of the lattice.

The plot of $\rho_s(T = 0)$ is shown in Fig.(5) for a range of K/J, we have set $J = 1/2$. The superfluid density curve has its maximum at $K/J = 0$, with a value of $\rho_s(T = 0) = 0.4059$ and decreases monotonically with increasing $|K|$ as one moves away from this maximum, in other words the ring exchange term decreases the value of $\rho_s(T = 0)$ from the pure XY result. On the positive-K side, ρ_s decreases relatively gradually, only becomes zero for extremely large values. This is consistent with the result obtained in the dispersion, which indicates that no soft modes develop for moderate values of positive K. Similar result was observed on a square lattice[8], which is consistent with quantum Monte Carlo simulation. On the negative-K side, the value of ρ_s decreases rapidly as it approches $K/J = -4/3$.

VI. SUPERFLUID DENSITY AND KOSTERLITZ-THOULESS TRANSITION AT FINITE TEMPERATURE

In this section we shall discuss the uniform superfluid phase in the $J-K$ model at finite temperatures. In order to calculate the finite temperature superfluid density, we replace the twisted ground state energy in (29) with the twisted free energy, that is

$$\rho_s(T \neq 0) = \frac{1}{N}\frac{\partial^2 F(\theta)}{\partial\theta^2}|_{\theta=0}. \quad (31)$$

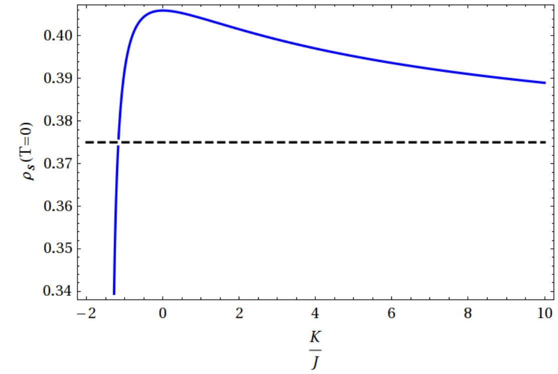

FIG. 5: (Color online) The superfluid density as a function of K/J. The dashed line is the mean-field result, $\rho_s^{MF} = 0.375$. The linear SW result for $K/J = 0$ is $\rho_s^{SW}(T = 0) = 0.4059$.

Using (26) we obtain

$$\begin{aligned}\rho_s(T \neq 0) \;=\; & \frac{3J}{4} + \frac{J}{2N}\sum_{\mathbf{k}}\Big[Q_{\mathbf{k}} - \frac{1}{\omega_{\mathbf{k}}} \quad (32)\\ & \cdot\;\left(A_{\mathbf{k}}Q_{\mathbf{k}} - B_{\mathbf{k}}S_{\mathbf{k}}\right)\left(1 + \frac{2}{e^{\epsilon_{\mathbf{k}}/T} - 1}\right)\Big].\end{aligned}$$

This expression for $\rho_s(T)$ is plotted as a function of T/J for several values of K/J (see Fig.(6)).

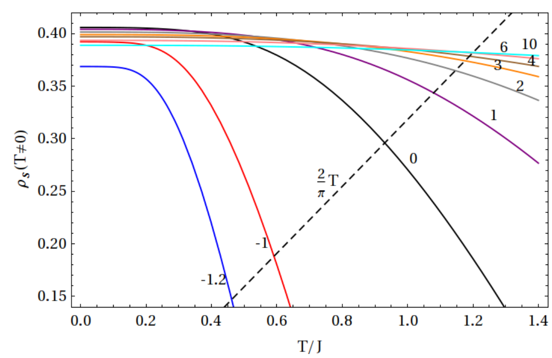

FIG. 6: (Color online) The superfluid density as a function of T/J. Lines are labelled by the parameter value K/J. The dashed line is the universal jump condition.

The graph shown in Fig.(6) is similar to the one obtained on a square lattice[8] which shows slowly decaying superfluid density. The dash line is the so-called *universal jump* condition

$$\frac{\rho_s(T_{KT})}{T_{KT}} = \frac{2}{\pi}, \quad (33)$$

which accounts for the discontinuity in $\rho_s(T)$. The estimate of T_{KT} can be found by solving $\rho_s(T)/T = 2/\pi$, using $\rho_s(T)$ from our spin wave theory. Using this procedure we found the KT transition temperature at $T_{KT} = 0.9295J$ for $K = 0$ and $J = 1/2$, the parameter values for the pure XY model. This is shown in Fig.(6),

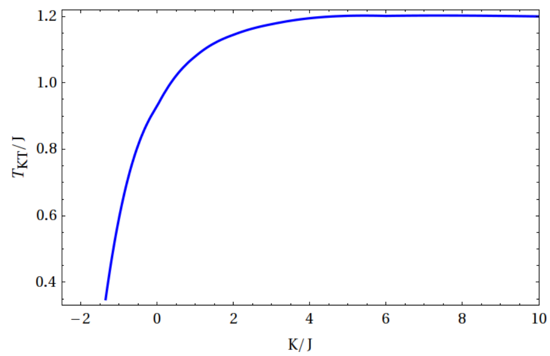

FIG. 7: (Color online) The KT transition phase boundary as calculated from linear SW theory. A maximum of $T_{KT}/J = 1.201$ occurs at approximately $K/J = 6.019$.

where the dashed line crosses the curve for $K/J = 0$. The plot of T_{KT} for non-zero values of K/J is shown in Fig.(7). It is interesting to see that the maximum does not occur at $K/J = 0$ but at $K/J \approx 6.019$ before dropping slowly. Similar to the case of square lattice[8], the KT transition drops to zero for extremely large value of K/J which cannot be captured by a simple spin wave theory. On the negative-K side, the phase boundary drops rapidly as K/J approaches $-4/3$.

VII. CONCLUSION

We have presented a comprehensive study of linear spin wave theory of hard-core bosons (zero field XY model) at half filling on the triangular lattice. We studied through linear spin wave theory the destruction of uniform superfluid phase in the bosonic ring exchange model on a triangular lattice at half filling. The dispersion of this model was calculated by applying the traditional Holstein-Primakoff representation and summing over the three plaquettes orientations on a triangular lattice. This calculation showed a spin wave instability and a development of three minima at $\mathbf{k} = \pm\mathbf{Q} = \left(\pm\frac{4\pi}{3}, 0\right)$, and $\mathbf{k} = \mathbf{Q} = (0,0)$ for $K/J = -4/3$. One should expect a phase transition from a superfluid phase to another phase at this wave vector $\mathbf{k} = \mathbf{Q} = \left(\frac{4\pi}{3}, 0\right)$ for $K/J = -4/3$. A more careful analysis of quantum Monte Carlo data should provide further insight into this issue.

The mean field superfluid density and ground state (zero temperature) spin wave superfluid density obtained in this model for $K/J = 0$ is bigger than that of a square lattice[8] despite the fact that both are $2D$ systems. This might be due to a larger number of nearest neighbours and plaquettes on a triangular lattice. We calculated the finite temperature uniform superfluid density and use it to estimate the Kosterlitz-Thouless transition temperature by forcing it to obey the universal quantum jump condition. We find that for $K<0$, the phase boundary monotonically decreases to $T = 0$ at $K/J = -4/3$, where a phase transition is expected. It has been shown with the Model Hamiltonian (2) on a square lattice using QMC simulations that for small $K>0$ away from the XY point, the zero-temperature spin stiffness value of the XY model is decreased[6]. Our results above seem to agree with this trend found in QMC simulations.

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